The time-dependent magnetohydrodynamic flow past a flat plate

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Two time-dependent magnetohydrodynamic flow problems are discussed. In Part I we consider the situation in which a semi-infinite flat plate is moved impulsively in its own plane into an electrically conducting viscous fluid. The ambient magnetic field has the same direction as the motion of the plate; it is found that when $\mu H_0^2/\rho U_0^2 < 1$, the flow pattern approaches asymptotically the steady flow found earlier (Greenspan & Carrier 1959). When $\mu H_0^2/\rho U_0^2 > 1$, the asymptotic state is one in which the fluid accompanies the plate in a rigid body motion as was anticipated in the earlier work.

In Part II, an infinite plate is moved impulsively in its own plane in the presence of an ambient magnetic field which is perpendicular to the plane of the plate. It is shown that the problem is not uniquely set until one specifies what three-dimensional problem reduces in the limit to the two-dimensional problem so defined. The answers in the conceptually acceptable limit case investigated here (the plate being a pipe of very large radius) have an asymmetry which at first sight is unexpected.

Part I. The Semi-infinite Plate Problem

1. Introduction

A recent paper (Greenspan & Carrier 1959) reports a study of the steady flow of a viscous, incompressible, electrically conducting fluid past a flat plate in the presence of a magnetic field. The unexpected result of that investigation which aroused our interest in the transient problem states that a steady flow with uniform velocity far from the plate can exist only when the ambient fluid speed is greater than the Alfven speed based on the ambient magnetic field (i.e. only when $\rho U_0^2/\mu H_0^2 > 1$). Here, we report the results of a study of the transient problem wherein the motion starts from rest at time zero. Specifically, we formulate the transient problem, linearize it using the modified Oseen technique, solve the linearized problem in a formal way, and, using approximation methods, find a relatively simple description of the flow which is valid for large time. The results are consistent with those of the steady-flow problem and no further surprises appear. The investigation, therefore, may be of more interest as an example of a generally applicable technique for extracting simple quantitative approximate descriptions of phenomena whose precise description is impeded by the details of a non-elementary 'Wiener-Hopf problem'.

2. The flow problem

The fundamental laws governing the flow of a conducting fluid in a magnetic field are Maxwell's equations and the conservation of mass and momentum. When the fluid is viscous and the speed is everywhere small compared to c (the speed of light), these equations take the form

$$\left.\begin{aligned} \mathbf{v}_{t'}(\mathbf{v}\,.\operatorname{grad})\,\mathbf{v} &= \rho^{-1}\operatorname{grad} p + \mu\rho^{-1}(\mathbf{j}\times\mathbf{H}) + \nu\Delta\mathbf{v}, \\ \operatorname{div}\mathbf{v} &= 0, \\ \operatorname{curl}\mathbf{H} &= \mathbf{j}, \quad \operatorname{div}\mathbf{H} = 0, \\ \operatorname{curl}\mathbf{E} &= -\mathbf{H}_{t'}, \quad \operatorname{div}\mathbf{E} = 0, \\ \mathbf{j} &= \sigma(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}). \end{aligned}\right\} \qquad (2.1)$$

Here $\mathbf{v}$ is the particle velocity, $\mathbf{H}$ the magnetic field, $\mathbf{E}$ the electric field, $\mathbf{j}$ the current density, σ the electrical conductivity, μ the permeability of the fluid, ρ the mass density, p the pressure and ν the kinematic viscosity.

We consider here the flow of such a fluid past the semi-infinite plate lying in $y = 0, x > 0$. The ambient magnetic field of intensity H_0 is uniform and is parallel to the x-axis. The plate is given a velocity U_0 in the negative x-direction at time zero and we wish to describe the ensuing flow field. If we regard this problem as the limit as $R \to \infty$ of another problem in which the plate is replaced by a pipe of radius R, the symmetry is such that no field quantities depend on the z co-ordinate. Note, however, that the current $\mathbf{j}$ and the electric field $\mathbf{E}$ will be directed in the z direction. The introduction of the vector potentials $\mathbf{v} = U_0 \operatorname{curl}\mathbf{k}\psi(x,y,t)$ and $\mathbf{H} = H_0\operatorname{curl}\mathbf{k}A(x,y,t)$ then allows us to simplify equations (2.1) to the form

$$\Delta\Delta\psi - \psi_y\Delta\psi_x + \psi_x\Delta\psi_y - \Delta\psi_t + \beta(A_y\Delta A_x - A_x\Delta A_y) = 0, \qquad (2.2)$$

$$\Delta A - \epsilon(\psi_y A_x + A_t - \psi_x A_y) = 0. \qquad (2.3)$$

In the foregoing $x = U_0 x'/\nu$, $y = U_0 y'/\nu$, $t = U_0^2 t'/\nu$, x', y', t', are the physical co-ordinates, $\beta = \mu H_0^2/\rho U_0^2$, $\epsilon = \mu\nu\sigma$, Δ is the Laplace operator in x, y, and all differential operations are taken with regard to the x, y, t co-ordinate system.

Equations (2.2) and (2.3) indicate that ψ and A are governed by a balance of diffusive and convective transport; it is known (Carrier & Lewis 1949; Greenspan & Carrier 1959), that in such problems the 'convective coefficients' ψ_y, ψ_x, A_y, A_x of (2.2) can be replaced by appropriately chosen averages. One rationalizes this by noting, for example, that the net convective effect in (2.2) of the terms $\psi_y\Delta\psi_x - \psi_x\Delta\psi_y \equiv u\Omega_x + v\Omega_y$ (where u, v is the velocity and Ω the vorticity) may be equivalent to a uniform horizontal convection at speed cU_0 where $0 < c < 1$. In the classical viscous-flow-past-a-flat-plate problem the choice $c = 0{\cdot}35$ gives remarkably accurate results. The character of the results is not affected by the choice of c, although the numbers which emerge do depend on c. Since our objective here is to determine the nature of the flow and its dependence on σ, μ, H_0, etc., we will not optimize the choice of these averages but will merely replace ψ_y and A_y in (2.2) and (2.3) by unity and replace ψ_x and A_x in (2.2) by zero. With this 'averaging' of the convective effects, (2.2) and (2.3) become

$$\Delta\Delta\psi - \Delta(\psi_x + \psi_t - \beta A_x) = 0, \qquad (2.4)$$

$$\Delta A - \epsilon(A_x + A_t - \psi_x) = 0. \qquad (2.5)$$

Note that any other choice for the averages of ψ_y, ψ_x, etc., would lead to equations which transform to (2.4) and (2.5) when the co-ordinates are scaled differently. This assures us that the nature of the results cannot be different for different choices of these quantities.

In view of the symmetry of the problem the boundary conditions require that $\psi(x,0,t) = A(x,0,t) = 0$, $\psi_y(x,0,t) = S(t)$ when $x > 0$, $\psi(x,y,t)$ and $A(x,y,t) \to 0$ as $\mathrm{Im}\,(x+iy)^{\frac{1}{2}} \to \infty$, and that $\psi_{yyy}(x,y,t)$ and $A_y(x,y,t)$ be continuous at $y = 0$.

If we define the Fourier–Laplace transforms of the potentials by

$$\overline{\psi}(\xi,\eta,s) = \int_0^\infty dt \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi x+\eta y)-st}\,\psi(x,y,t)\,dx\,dy, \tag{2.6}$$

the conventional transformation of (2.4) and (2.5) yields

$$(\eta^2+\xi^2)\,[\{\eta^2+(\xi^2+i\xi+s)\}\,\overline{\psi}-i\beta\xi\overline{A}] = i\eta\bar{f}(\xi,s), \tag{2.7}$$

$$[\eta^2+\xi^2+\epsilon(i\xi+s)]\,\overline{A}-i\epsilon\xi\overline{\psi} = 0, \tag{2.8}$$

where $f(x,t)$ (whose transform in x and t is $\bar{f}$) is given by

$$f = \psi_{yy}(x,0+,t)-\psi_{yy}(x,0-,t).$$

We solve (2.7) and (2.8) to obtain

$$\overline{\psi}(\xi,\eta,s) = \frac{i\eta\bar{f}(\xi,s)\,[\xi^2+\eta^2+\epsilon(i\xi+s)]}{\{(\xi^2+\eta^2+i\xi+s)\,(\xi^2+\eta^2+\epsilon[i\xi+s])+\beta\epsilon\xi^2\}\,(\xi^2+\eta^2)}, \tag{2.9}$$

and we define

$$\psi^*(\xi,y,s) = (\tfrac{1}{2}\pi)\int_{-\infty}^{\infty} e^{i\eta y}\,\overline{\psi}(\xi,\eta,s)\,d\eta. \tag{2.10}$$

Thus, ψ^* is the transform in x and t only of $\psi(x,y,t)$. If we perform the integration of (2.10), we obtain†

$$-2\psi^*(\xi,y,s) = \bar{f}(\xi,s)\,\{(\eta_3^2-\xi^2)\,(\eta^2-\eta_2^2)\,e^{-|\xi|\,|y|}+(\eta_3^2-\eta^2)\,(\eta_2^2-\xi^2)\,e^{-\eta|y|}$$
$$+(\eta_3^2-\eta_2^2)\,(\xi^2-\eta^2)\,e^{-\eta_2|y|}\}\,[(\eta_1^2-\eta_2^2)\,(\eta_1^2-\xi^2)\,(\eta_2^2-\xi^2)]^{-1}, \tag{2.11}$$

where $\eta_3^2 = \xi^2+i\xi+s$ and $2\eta_1^2$,

$$2\eta_2^2 = 2\xi^2+(1+\epsilon)\,(i\xi+s)\pm[(1-\epsilon)^2\,(i\xi+s)^2+\beta\epsilon(i\xi)^2]^{\frac{1}{2}}.$$

In particular,

$$2\psi_y^*(\xi,0,s) = \bar{f}\{|\xi|\,(\eta_3^2-\xi^2)\,(\eta_1^2-\eta_2^2)+\eta_1(\eta_3^2-\eta_1^2)\,(\eta_2^2-\xi^2)$$
$$+\eta_2(\eta_3^2-\eta_2^2)\,(\xi^2-\eta^2)\}\,[(\eta_1^2-\eta_2^2)\,(\eta_1^2-\xi^2)\,(\eta_2^2-\xi^2)] = \bar{f}\overline{K}(\xi,s). \tag{2.12}$$

When $\epsilon = 1$, (2.12) reduces to

$$2\psi_y^*(\xi,0,s)|_{\epsilon=1} = -\tfrac{1}{2}\bar{f}[\{|\xi|+[\xi^2+i(1+\beta)\,\xi+s]^{\frac{1}{2}}\}^{-1}+\{|\xi|+[\xi^2+i(1-\beta)\,\xi-s]^{\frac{1}{2}}\}^{-1}]$$
$$= -\bar{f}\overline{K}_1(\xi,s); \tag{2.13}$$

and when $\epsilon \to \infty$ (infinite conductivity)

$$2\psi_y^*(\xi,0,s) \to -\bar{f}[|\xi|+[\xi^2+i\xi+s-\beta\xi^2/(i\xi+s)]^{\frac{1}{2}}]^{-1} = -\bar{f}\overline{K}_\infty(\xi,s). \tag{2.14}$$

Equations (2.13) and (2.14) define $\overline{K}_1$ and $\overline{K}_\infty$.

† This calculation reduces to the evaluation of some residues.

The function $\bar{f}(\xi)$ of (2.12) can be found, in principle, by computing $\bar{K}_+(\xi, s)$ and $\bar{K}_-(\xi, s)$, the two factors of $\bar{K}(\xi, s)$ which are analytic functions of ξ in overlapping upper and lower half planes.† Such factors exist provided $|\xi|$ is replaced by $\sqrt{(\xi^2+m^2)}$ and the limit $m \to 0$ is taken subsequently. Since $\psi_y(x, 0, t)$ is unity for $x > 0$, $t > 0$, and is not known for $x < 0$, $t > 0$, we may write

$$\psi_y(x, 0, t) = S(t)\, S(x) + V(x, t), \tag{2.15}$$

where V is zero when either $t < 0$ or $x > 0$. It follows that

$$\psi_y^*(\xi, 0, s) = (i\xi s)^{-1} + \bar{V}(\xi, s),$$

where $\bar{V}$ is analytic in ξ in some upper half plane whenever $s > 0$. Equation (2.12) can now be written

$$2([i\xi s]^{-1} + V) = \bar{f}\bar{K}_+\bar{K}_-. \tag{2.16}$$

The usual arguments of the Wiener–Hopf method lead to the result

$$\bar{f} = 2/[\bar{K}_+(0, s)\, \bar{K}_-(\xi, s)\, i\xi s], \tag{2.17}$$

and the inversion of $\bar{f}$ over ξ and s completes the calculation of $f(x, t)$. Since f is twice the skin friction on each side of the plate, the character of the flow field can be inferred directly; if more details are wanted, $\psi^*(\xi, y, s)$ as defined in (2.11) must be inverted to obtain $\psi(x, y, t)$. The calculation alluded to here is not very useful because $\bar{K}_+$ and $\bar{K}_-$ are such exceedingly messy functions that no relatively straightforward description of $f(x, t)$ can be so obtained. Since our objective is an interpretable description we shall obtain one at a minor sacrifice in accuracy. Equation (2.12) as well as its special forms (2.13) and (2.14) are each equivalent to an integral equation

$$\psi_y^*(x, 0, s) = \int_0^\infty f^*(x', s)\, K(x-x', s)\, dx', \tag{2.18}$$

where ψ_y^* and f^* are the transforms of ψ_y and f with regard to t only and K is the function whose x transform is $\bar{K}(\xi, s)$. If the kernel $K(x, s)$ of (2.18) is replaced by a suitable substitute kernel, $N(x, s)$, the solution of the so modified integral equation cannot differ markedly from that of (2.18). In particular, if $N(x, s)$ and $K(x, s)$ have the same area, $\int_0^\infty K(x, s)\, dx$, the same first moment, $\int_0^\infty xK(x, s)\, dx$, the same singularity at $x = 0$, and if $K(x, s)/N(x, s)$ tends uniformly to unity as $s \to 0$ the solutions of these integral equations will have identical gross features for $t \gg 1$. A more comprehensive argument on this point will be published elsewhere.‡ It is simpler to choose the substitute kernel, N, for special cases first; when $\beta \leqslant 1$, the case $\epsilon = 1$ is an especially convenient one. A suitable choice for $N(x, s)$ is best described in terms of $\bar{N}(\xi, s)$; in fact, that $\bar{N}(\xi, s)$ for which each of the foregoing considerations is met and which is easily factored§ is given by

$$2\bar{N}(\xi, s) = (1 - i\alpha\xi)^{-\frac{1}{2}}[\{s + i(1+\beta)\,\xi\}^{-\frac{1}{2}} + \{s + i(1-\beta)\,\xi\}^{\frac{1}{2}}], \tag{2.19}$$

where

$$\alpha = [(1+\beta)^{-\frac{1}{2}} + (1-\beta)^{-\frac{1}{2}}]^2.$$

† The details of the Wiener–Hopf process are deliberately omitted here but may be found, for example, in Greenspan & Carrier, 1959.

‡ The original suggestion that this substitution is useful is due to Koiter (1954). The method of choosing the kernel is that of Carrier (1959).

§ N would have no advantage over K if it were not easily factored.

Equation (2.17) now gives

$$\bar{f} = 2[\{s+i(1+\sqrt{\beta})\,\xi\}^{-\frac{1}{2}} + \{s+i(1-\sqrt{\beta})\,\xi\}^{-\frac{1}{2}}]\, i\xi s \tag{2.20}$$

and this can be inverted explicitly to obtain

$$\begin{aligned} f(x,t) = {} & \frac{2(1+\beta)}{\beta}\{\sqrt{[(1-\beta)/\pi x]}\, S[(1-\sqrt{\beta})\,t-x] + 1/\sqrt{(\pi t)}\, S[x-(1-\sqrt{\beta})\,t]\} \\ & \frac{-2(1-\beta)}{\beta}\{\sqrt{[(1+\beta)/\pi x]}\, S[(1+\sqrt{\beta})\,t-x] + (\pi t)^{-\frac{1}{2}}\, S[x-(1+\sqrt{\beta})\,t]\} \\ & + (1/\beta\sqrt{\pi}\, t^{\frac{3}{2}})\,\{[x-(1-\sqrt{\beta})\,t]\, S[x-(1-\sqrt{\beta})\,t] \\ & [x-(1+\sqrt{\beta})\,t]\, S[x-(1+\sqrt{\beta})\,t]\}. \end{aligned} \tag{2.21}$$

Simply stated, (2.21) implies that: For $0 < x < (1-\sqrt{\beta})\,t$ the skin friction is precisely that of the steady-state problem; further downstream, for $x > (1+\sqrt{\beta})\,t$ in fact, the skin friction is that which an infinite plate would experience. In $(1-\sqrt{\beta})\,t < x < (1+\sqrt{\beta})\,t$ a smooth transition occurs. Note that when $\beta < 1$, both Alfven waves (the one going with the fluid and that going against the fluid) go downstream relative to the plate. Behind the slower one the steady state prevails and ahead of the faster the fluid does not know the plate has a front edge. These statements would be mildly modified by the diffusive effects that are suppressed by our substitution of N for K.

The corresponding approximations can be written down for all ϵ. The results are intrinsically the same as those for $\epsilon = 1$; in $x < (1-\sqrt{\beta})\,t$ the steady flow prevails, in $x > (1+\sqrt{\beta})\,t$ the infinite plate flow occurs. Since v and **H** are precisely parallel in the infinite plate problem, **H** plays no role in determining the flow field in $x > (1+\sqrt{\beta})\,t$. In $(1-\sqrt{\beta})\,t < x < (1+\sqrt{\beta})\,t$ the foregoing regions are joined smoothly.

When $\beta > 1$, an Alfven wave can travel upstream relative to the plate and the phenomenon is very different from that associated with $\beta < 1$. In this instance the situation in which $\epsilon \to \infty$ provides the simplest example. The K_∞ of (2.14) is replaced by

$$\bar{N}_\infty = \{(i\xi+s)/[i\xi+s/(1-\sqrt{\beta})]\,[i\xi+s/(1+\sqrt{\beta})]\,[1-\beta-4i\xi]\}^{\frac{1}{2}}. \tag{2.22}$$

When $\beta < 1$, this substitution is consistent with the foregoing results, but when $\beta > 1$, $\bar{N}_+$ becomes $i\xi+s(1-\sqrt{\beta})^{-\frac{1}{2}}$ instead of $[1-\beta-4i\xi]^{-\frac{1}{2}}$, its value when $\beta < 1$. In this case $\bar{f}(\xi, s)$ is given by

$$\bar{f} = 2[\{s/(1-\sqrt{\beta})\}\,\{i\xi+s/(1+\sqrt{\beta})\}\,(1-\beta 4i\xi)/(i\xi+s)]^{\frac{1}{2}}/i\xi s. \tag{2.23}$$

The function $f(x,t)$ obtained by inverting $\bar{f}$ tends to zero at a rate governed by the factor $t^{-\frac{1}{2}}$. Thus, the prediction that the only steady flow for $\beta > 1$ is one in which the fluid and plate undergo a rigid body motion, is substantiated.

Again the same type of approximation can be chosen for all ϵ and the same result emerges. That is, $f \to 0$ like $t^{-\frac{1}{2}}$ for all ϵ. Note that if $f \to 0$ as $t \to \infty$, $\psi_y(x,y,t)$ must tend to -1 for all x, y as $t \to \infty$.

PART II. RAYLEIGH MOTION IN A MAGNETIC FIELD

3. Introduction

One of the classical problems of viscous fluid theory whose study has aided our understanding of the dynamics of such fluids is that in which an infinitely extended thin plate is immersed in an unbounded viscous fluid and, at time zero, is suddenly given a constant velocity in its own plane. There are many magnetohydrodynamic extensions of this problem which can be defined; one of these which is informative and which provides a few surprises is the following.

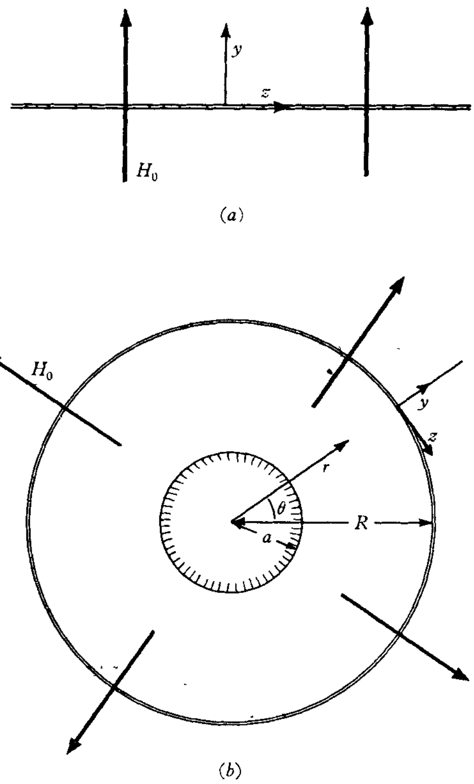

FIGURE 1. The geometry of the cylindrical problem and its limit, the plate problem.

Let there be a uniform ambient magnetic field perpendicular to the plane of the plate and let the geometry be considered as the limit as $R \to \infty$ of a situation wherein a pipe of radius R is exposed to such an ambient magnetic field and receives a step-function axial velocity at time zero. These geometries and the co-ordinate systems to be used are shown in figure 1. The use of this particular limiting process allows us to specify that the field quantities are dependent only on the time and on r, the co-ordinate perpendicular to the plate. In particular, since no electric charge can accumulate because of the axial symmetry, the electric field intensity, $\mathbf{E}$, must become zero when the steady state has been achieved (i.e. after t has been allowed to become infinite). If this problem is solved for the geometry of figure 1 *a*, an answer can be obtained, but after taking the limit $t \to \infty$ it is readily seen that the velocity distribution is of the form $A + Be^{-b|y|}$, where A, B and b, depend on the physical parameters of the problem, **E** is not zero, and the solution is not interpretable in terms of the postulated problem. This solution corresponds, in fact, to a situation in which charge is allowed to accumulate in a very artificial manner. It appears then, that one must solve the cylindrical problem and investigate, carefully, the behaviour as $R \to \infty$. Since the solution to the cylindrical problem contains the description of the solution of the plate problem mentioned above, we shall not develop the plate solution separately. Specifically, then, we shall treat the geometry of figure 1*b* where the solid core of radius a is fixed in space and has conductivity σ_1 and permeability μ_1; the fluid in the annular region $a < r < R$ is identical with that in $r > R$ and has conductivity σ, permeability μ, kinematic viscosity ν, and density ρ; the indefinitely thin solid sheet at R, whose electrical properties do not influence the problem, has the motion $US(t)$ in the axial direction. The fluid is at rest, **E** is zero and $\mathbf{H} = \hat{r}H_0R/r$ until time zero. The ambient magnetic field adopted here could not be produced precisely in an experiment but neither could the uniform fields of infinite extent which we postulate in other problems whose study improves our understanding of these magneto-hydrodynamic phenomena.

4. Analysis of the cylindrical problem

The physical laws governing the phenomenon in $r > R$ are again given by (2.1). We introduce cylindrical co-ordinates r, θ, z, and define the following dimensionless quantities,

$$\eta = Ur/\nu, \quad \tau = U^2t/\nu, \quad \mathbf{H} = H_0[\hat{\mathbf{r}}(R/r) + \hat{\mathbf{z}}h], \quad \mathbf{E} = \mu H_0 U\phi\hat{\boldsymbol{\theta}}, \quad \mathbf{v} = Uu\hat{\mathbf{z}},$$
$$k = RU/\nu, \quad b = aU/\nu, \quad \beta = \mu H_0^2/\rho U_0^2, \quad \epsilon = \mu\nu\sigma.$$

The equations then become

$$(\eta u_\eta)_\eta + \beta k h_\eta - \eta u_\tau = 0, \tag{4.1}$$
$$\eta h_\eta + \epsilon k u + \epsilon\eta\phi = 0, \tag{4.2}$$
$$(\eta\phi)_\eta + \eta h_\tau = 0. \tag{4.3}$$

In the core, since the properties are different from those of the fluid and since the velocity is zero, we have, instead of (4.2) and (4.3),

$$(\eta h_\eta)_\eta - \lambda\eta h_\tau = 0, \tag{4.4}$$

and

$$\phi = -(\sigma/\sigma_1)h_\eta, \tag{4.5}$$

where

$$\lambda = \mu_1\sigma_1/\mu\sigma.$$

The solutions of (4.1), (4.2) and (4.3) are elementary only when $\epsilon = 1$. For other ϵ, the behaviour of the system is different only in detail; we shall indicate the nature of this distinction later, but will develop only the case $\epsilon = 1$.

With $\epsilon = 1$ then, the Laplace transforms of the solutions of (4.1), (4.2) and (4.3), which vanish as $\eta \to \infty$ can be represented in the form

$$\bar{u}(\eta, s) = A(\eta/k)^m K_m(\eta\sqrt{s}) + B(\eta/k)^{-m} K_m(\eta\sqrt{s}), \tag{4.6}$$
$$\bar{h}(\eta, s) = -A(\eta/k)^m K_m(\eta\sqrt{s}) + B(\eta/k)^{-m} K_m(\eta\sqrt{s}), \tag{4.7}$$
$$\bar{\phi}(\eta, s) = -\bar{h}_\eta - k\bar{u}/\eta, \tag{4.8}$$

where $m = k\sqrt{\beta}/2$.

In the annular region, $b < \eta < k$, the fields are given by (4.8) and by

$$\bar{u}(\eta, s) = C(\eta/k)^m K_m(\eta\sqrt{s}) + D(\eta/k)^m I_m(\eta\sqrt{s}) + E(\eta/k)^{-m} K_m(\eta\sqrt{s}) + F(\eta/k)^{-m} I_m(\eta\sqrt{s}), \quad (4.9)$$

$$\bar{h}(\eta, s) = -C(\eta/k)^m K_m(\eta\sqrt{s}) - D(\eta/k)^m I_m(\eta\sqrt{s}) + E(\eta/k)^{-m} K_m(\eta\sqrt{s}) + F(\eta/k)^{-m} I_m(\eta\sqrt{s}). \quad (4.10)$$

In the core:
$$\bar{h} = G I_0(\eta\sqrt{[\lambda s]}), \quad (4.11)$$

and
$$\bar{\phi} = -\sqrt{(\omega s)}\, G I_1[\eta\sqrt{(\lambda s)}], \quad (4.12)$$

where $\omega = \sigma\mu_1/\sigma_1\mu$. The boundary conditions require that

$$\bar{u}(k+, s) = 1/s, \quad \bar{h}(k+, s) = \bar{h}(k-, s), \quad \bar{\phi}(k+, s) = \bar{\phi}(k-, s),$$

$$\bar{u}(k+, s) = \bar{u}(k-, s), \quad \bar{u}(b, s) = 0, \quad \bar{h}(b+, s) = \bar{h}(b-, s), \quad \bar{\phi}(b+, s) = \bar{\phi}(b-, s).$$

Using (4.6) to (4.11) inclusive, these boundary conditions become, in matrix form (with unchanged order),

$$M_{ij} A_j = \delta_{i1} s^{-1}, \quad (4.13)$$

where A_i is the vector $(A, B, \ldots, G)$ and M_{ij} is the following matrix.

$$\begin{matrix}
K_m(k\sqrt{s}) & K_m(k\sqrt{s}) & 0 & 0 & 0 & 0 & 0 \\
-K_m(k\sqrt{s}) & K_m(k\sqrt{s}) & K_m(k\sqrt{s}) & I_m(k\sqrt{s}) & -K_m(k\sqrt{s}) & -I_m(k\sqrt{s}) & 0 \\
K_{m-1}(k\sqrt{s}) & -K_{m+1}(k\sqrt{s}) & -K_{m-1}(k\sqrt{s}) & I_{m-1}(k\sqrt{s}) & K_{m+1}(k\sqrt{s}) & I_{m+1}(k\sqrt{s}) & 0 \\
-K_m(k\sqrt{s}) & -K_m(k\sqrt{s}) & K_m(k\sqrt{s}) & I_m(k\sqrt{s}) & K_m(k\sqrt{s}) & I_m(k\sqrt{s}) & 0 \\
0 & 0 & \left(\frac{b}{k}\right)^m K_m(b\sqrt{s}) & \left(\frac{b}{k}\right)^m I_m(b\sqrt{s}) & \left(\frac{b}{k}\right)^{-m} K_m(b\sqrt{s}) & \left(\frac{b}{k}\right)^{-m} I_m(b\sqrt{s}) & 0 \\
0 & 0 & -\left(\frac{b}{k}\right)^m K_m(b\sqrt{s}) & -\left(\frac{b}{k}\right)^m I_m(b\sqrt{s}) & \left(\frac{b}{k}\right)^{-m} K_m(b\sqrt{s}) & \left(\frac{b}{k}\right)^{-m} I_m(b\sqrt{s}) & -2\sqrt{\beta}\, I_0(b\sqrt{s}) \\
0 & 0 & \left(\frac{b}{k}\right)^m K_{m-1}(b\sqrt{s}) & -\left(\frac{b}{k}\right)^m I_{m-1}(b\sqrt{s}) & -\left(\frac{b}{k}\right)^{-m} K_{m+1}(b\sqrt{s}) & \left(\frac{b}{k}\right)^{-m} I_{m+1}(b\sqrt{s}) & -2\sqrt{\beta}\sqrt{\omega}\, I_1(b\sqrt{s})
\end{matrix}$$

It is especially convenient to represent each of the field quantities by the ratio of two determinants. In particular, the velocity in the outer fluid is given by

$$\bar{u}(\eta, s) = M^*/M, \quad (4.14)$$

where M is the determinant of M_{ij} and M^* is the same determinant with the first row replaced by the entries $[s^{-1}(\eta/k)^m K_m(\eta\sqrt{s}), s^{-1}(\eta/k)^{-m} K_m(\eta\sqrt{s}), 0, 0, 0, 0, 0]$. The numerator determinant for the velocity in the annulus would have a top row in which the third to sixth entries inclusive are $1/s$ times the factors of C, D, E, F, in (4.9). Equally simple forms obtain for each of the other quantities of interest. However, the useful information can all be deduced for (4.14). It is evident that the direct inversion of M^*/M would be very difficult; it is therefore profitable to consider the behaviour of M^*/M for very large k. To do this we divide the columns of both M^* and M by $M_{11}, M_{12}, M_{23}, M_{24}, -M_{25}, -M_{26}, M_{27}$, respectively. We then examine each entry and replace it by its limit as $k \to \infty$ only if that limit is valid for *all* non-negative s. It is especially important to avoid replacement when the limit sequence $s \to 0$, $k \to \infty$, provides a different result from that of the sequence $k \to \infty$, $s \to 0$. The significance of this is more easily understood when the results have been obtained.

Fortunately, asymptotic representations of $K_m(z)$ and $I_m(z)$, valid for large m

and for *all* real non-negative z are readily available and we can list the pertinent information. In particular (with $\eta = k+y$, $y \ll k$, $b \ll k$),

$$\frac{K_m[(k+y)\sqrt{s}]}{K_m(k\sqrt{s})} \sim \exp\{-y(s+\tfrac{1}{4}\beta)^{\frac{1}{2}}\},$$

$$(b/k)^m \frac{K_m(b\sqrt{s})}{K_m(k\sqrt{s})} \sim (1+4s/\beta)^{\frac{1}{4}}$$
$$\times \exp(-m[\log\{\tfrac{1}{2}+\tfrac{1}{2}(1+4s/\beta)^{\frac{1}{2}}\}+1-(1+4s/\beta)^{\frac{1}{2}}]) \equiv g(m,s),$$

$$(b/k)^{-m}\frac{I_m(b\sqrt{s})}{I_m(k\sqrt{s})} \sim g(-m,s),$$

$$\frac{K_{m+1}(k\sqrt{s})}{K_m(k\sqrt{s})} \sim \frac{(4s/\beta)^{\frac{1}{2}}}{1+(1+4s/\beta)^{\frac{1}{2}}} = f(s),$$

$$\frac{K_{m-1}(k\sqrt{s})}{K_m(k\sqrt{s})} \sim 1/f(s).$$

The use of these allows us to evaluate (4.14) and obtain

$$u(\eta,s) = \frac{T}{s}\exp\{-y[(s+\tfrac{1}{4}\beta)^{\frac{1}{2}}-\tfrac{1}{2}\beta^{\frac{1}{2}}]\}$$

where
$$T = \left[1-\frac{(1-e^{-\sqrt{\beta}y})(f^2+1)}{2-(f^2+1)g(-m,s)/g(m,s)}\right].$$

For real non-negative s, T can be expanded to give

$$T = 1-\frac{1}{2}\left\{\sum_{n=0}^{\infty}[(f^2+1)g(-m,s)/2g(m,s)]^n\right\}(1-e^{-y\sqrt{\beta}})(f^2+1)$$
$$= \tfrac{1}{2}(1+e^{-y\sqrt{\beta}})-\tfrac{1}{2}(1-e^{-y\sqrt{\beta}})\sum_{n=1}^{\infty}[\tfrac{1}{2}g(-m,s)/g(m,s)]^n+Q(\eta,s).$$

Here $Q(\eta,s)$ is essentially linear in s for small s.

This series can be inverted term by term and the first term is precisely

$$u_0(\eta,\tau) = \tfrac{1}{4}(1+e^{-y\sqrt{\beta}})\left[\operatorname{erfc}\left(\frac{y-\tau\sqrt{\beta}}{2\sqrt{\tau}}\right)+e^{y\sqrt{\beta}}\operatorname{erfc}\left(\frac{y+\tau\sqrt{\beta}}{2\sqrt{\tau}}\right)\right].$$

The subsequent terms are most readily estimated (with great accuracy for large 'time' τ) by the method of steepest descent. For example, the next term is

$$u_1(\eta,\tau) = \frac{(1-e^{-y\sqrt{\beta}})}{8\pi}\int_{-i\infty}^{i\infty}\exp[s\tau-(y+k)\{(s+\tfrac{1}{4}\beta)^{\frac{1}{2}}-\tfrac{1}{2}\beta^{\frac{1}{2}}\}+O(ks^2)]s^{-1}\,ds.$$

When the saddle point of the exponent lies to the right of the origin, the integral is of order $\tau^{-\frac{1}{2}}$; when it lies to the left of the origin the integral is of order unity. Furthermore, for $\tau\sqrt{\beta} = y+k$ the saddle point is at the origin, thus, the mid-point of the transition lies at $y+k = \tau\sqrt{\beta}$. In fact,

$$u_1(k+y,\tau) \sim \tfrac{1}{8}(1-e^{-y\sqrt{\beta}})\operatorname{erfc}\left(\frac{y+k-\tau\sqrt{\beta}}{2\sqrt{\tau}}\right).$$

Thus, like u_0, this term represents a wave whose propagation speed c (in dimensional form) is the Alfven speed; however, this wave has an apparent starting time which is $t_1 = R/c$ (i.e. $\tau_1 = k/\sqrt{\beta}$). This is precisely the time required for the transmission of a wave from the pipe to the origin and back to the pipe.† Thus, the u_1 term represents a wave which has reflected from the core after initiation at

† Note that the time to travel from $r = 0$ to $r = R$ in the ambient field $H = H_0R/r$ is exactly half the time required to go the same distance in a uniform field H_0.

the pipe and has arrived late at y. The next term in the series will give another wave, whose history involves transmission to the core, reflexion back to the pipe, a return to the core, and finally transmission to y. This term is just half as big as u_1 and, in fact, all subsequent terms decrease in size as a geometric series. The contribution of Q is quite different since the integrand associated with Q is not singular at the origin or anywhere on the imaginary axis of the s-plane. There can be no contributions of the foregoing type and, in fact, Q merely yields shape corrections to the u_n. Each of these corrections tends to zero as $t \to \infty$ (see figure 2).

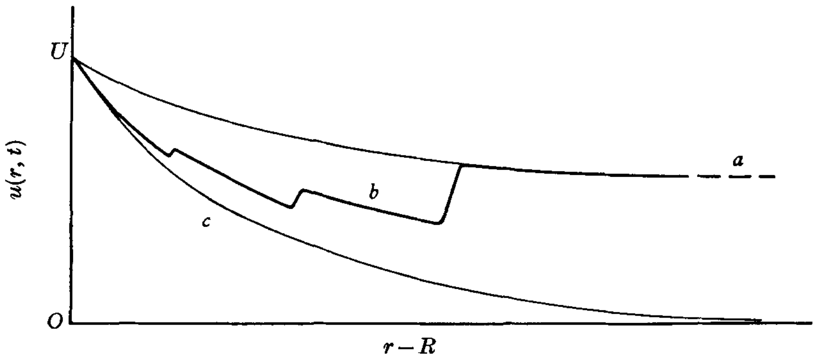

FIGURE 2. Curve a, $u_0(r, \infty)$; curve b, $u(r, t)$; curve c, $u(r, \infty)$.

If we had let $k \to \infty$ with $s \neq 0$ in the $g(m, s)$ terms, we would have obtained $u \equiv u_0$. From a physical viewpoint this means that we would have suppressed the reflexions from the core; such a suppression would imply that we had not solved a cylindrical problem, but had studied some more artificial situation. In fact, u_0 is the solution one obtains by a direct attack on the flat plate problem discussed in the introduction. Note that $2u_0(y, \infty) = 1 + e^{-y\sqrt{\beta}}$ and that $u_0 \to \frac{1}{2}$ as $y \to \infty$; it follows that $\mathbf{E}(y, \infty)$ cannot be zero and direct calculation of $\mathbf{E}$ confirms this.

On the other hand, if we let $t \to \infty$ first, the rigorous inversion of $\bar{u}$ gives

$$u(\eta, \infty) = e^{-y\sqrt{\beta}}.$$

Thus, the velocity vanishes far from the plate, $\mathbf{E}$ *can* be zero, and no physical requirements are violated.

When $\epsilon \neq 1$, the analysis is more difficult. However, the nature of the result can be deduced by noting that the first wave to arrive at a position y in the cylindrical problem would be very closely related to the direct solution of the flat plate problem for the same ϵ. This flat plate solution describes a wave which differs from that of the $\epsilon = 1$ case in that the velocity u_0 behind the wave front has the form

$$u_0 \sim a + (1-s)\, e^{-\gamma y},$$

where $a = \sqrt{\epsilon}/(1+\sqrt{\epsilon})$, $\gamma = \sqrt{(\epsilon\beta)}$.

In the cylindrical problem, the first reflexion will replace part of the first term of U_0 by a further $e^{-\gamma y}$ contribution and the entire wave sequence will lead to a velocity distribution (for large R)

$$u(\eta, \infty) \sim e^{-\gamma y}.$$

The quantitative description of these waves could be obtained by appealing directly to asymptotic techniques for solving (4.1) to (4.3) for large k; however, it is not clear that the further clarification so achieved would justify the labour involved.

Perhaps the most useful lesson to be drawn from the foregoing is the reminder that the replacement of conceptually acceptable three-dimensional problems by more tractable two-dimensional ones must be carefully justified. The specification of the manner in which current paths are closed and the implications regarding the induced fields must be consistent with the formulation of the two-dimensional problem.

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